JOURNAL OF MATERIALS SCIENCE **33** (1998) 4667–4669

The nucleation of a crack at the surface of a circular cylindrical cavity

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From the basis of a cohesive zone description, the paper formulates a criterion for the nucleation of a crack at the surface of a circular cylindrical cavity in an infinite solid. The criterion is expressed in terms of the internal pressure within the cavity, the stress system in the absence of the cavity, the cavity size and the cohesive-zone material characteristics. With attention being focussed on the situation where the cohesive-zone size at crack nucleation is small compared with the cavity size, the nucleation criterion is expressed in a particularly simple form.

1. Introduction

Tensile-type fracture at the surface of a cavity in a quasi-brittle material is important with regard to fracture events associated with boreholes, tunnels or mining-related underground cavities. The classic Griffith theory [1] can be applied to the propagation of a crack, but this theory is not appropriate to the nucleation of a tensile-type fracture at the surface of a cavity, which is the subject matter of the paper's considerations. With a quasi-brittle material, it has been observed experimentally that fracture nucleation involves the formation of a damage zone that is able to grow stably until cohesion is completely lost within the zone (at the cavity surface in the cavity situation).

The simplest way of representing the damage zone is to use the cohesive zone description, whereby a single infinitesimally thin two-dimensional cohesive zone starts to form at the cavity surface when the tensile stress at the surface attains some critical value p_c, and as the loadings increase, the zone spreads away from the surface. The zone can be characterised by a material-specific relation between the tensile stress (p) acting across the zone and the relative displacement (v) between the zone faces, with p being a maximum (with value p_c) at the leading edge of the cohesive zone. The stress p decreases as the displacement v increases and p falls to zero at the trailing edge of the zone when the displacement v attains a critical value v_c. There is then a complete loss of cohesion and crack nucleation is said to occur. It is possible to quantify the nucleation event for any prescribed cohesive zone softening law using numerical methods (see for example the work of Hashida and co-workers [2] on granite).

However, to simplify the considerations so that we can use analytical procedures—thereby allowing us to clearly see the interplay between material, geometrical and stress parameters,—it will be assumed that the stress p within the cohesive zone remains constant at the value p_c until the displacement v attains the critical value v_c when p is assumed to fall abruptly from p_c to zero. This is the classic DBCS (Dugdale-Bilby-Cottrell-Swinden) representation [3, 4] that is frequently used to model stress-relaxation phenomena. When this representation is applied to the cavity problem, crack nucleation occurs when the displacement v_T at the trailing edge of the cohesive zone, i.e. the cavity surface, attains the critical value v_c.

The cohesive-zone description, coupled with the DBCS representation, is used in this paper to quantify crack nucleation at the surface of a cavity, with attention being focussed on the situation where the cohesive-zone size at crack nucleation is small compared with the cavity size, since this allows the nucleation criterion to be expressed in a particularly simple form.

2. Theoretical analysis

Fig. 1 shows the model of an infinite solid, subjected to external stresses $p_{11} = -\sigma_1$ and $p_{22} = -\sigma_2$. This solid contains a circular cylindrical cavity of radius a and with its axis in the three o'clock direction, the internal pressure within the cavity being p_I. There are cohesive zones, within which the tensile stress is p_c. They emanate from the cavity surface at the three o'clock and nine o'clock positions. It is assumed either that these are positions of maximum circumferential tensile stress at the cavity surface, or that planes of weakness coincide with these positions.

The stress distribution in the absence of the cohesive zones is given by expressions due to Kinch [5] and reported recently by Atkinson and Thiercelin [6]. The circumferential tensile stress at the cavity surface is given by the expression

$$p_{\theta\theta}(r = a) = -(\sigma_1 + \sigma_2) + 2(\sigma_1 - \sigma_2)\cos 2\theta + p_I \qquad (1)$$

and it is immediately seen that, with $\sigma_1 > \sigma_2$ and with σ_1 and σ_2 both assumed to be positive, the circumferential

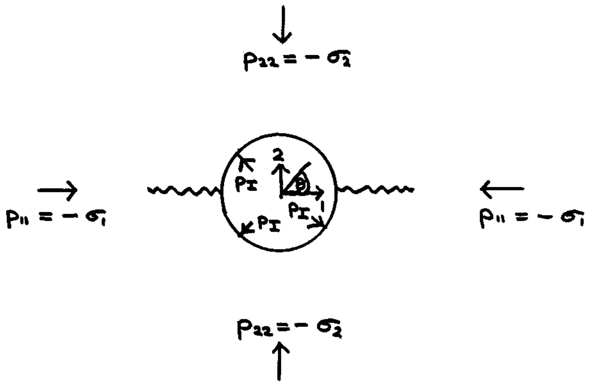

Figure 1 The model of a circular cylindrical cavity in an infinite solid.

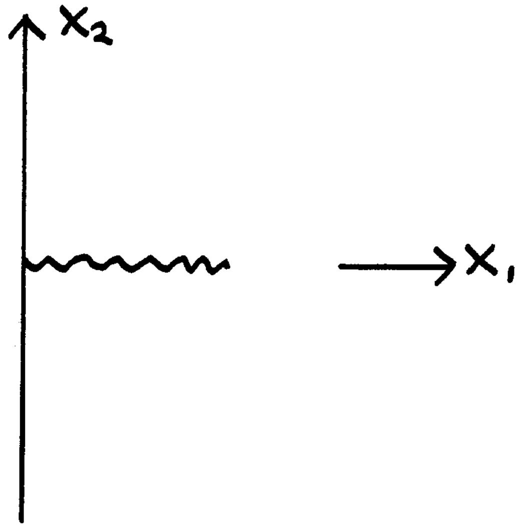

Figure 2 The planar-surface model.

tensile stress at the cavity surface is a maximum (with value $\sigma_2 - 3\sigma_1 + p_{\mathrm{I}}$) at the three o'clock and nine o'clock positions and fracture will be favoured at these positions if there is no other preferential plane of weakness. The tensile stress p_{22} along the plane $x_2 = 0$ at a distance x ahead of the cavity surface is given by the expression

$$p_{22}(x) = -\frac{(\sigma_1 + \sigma_2)}{2}\left[1 + \frac{a^2}{(a+x)^2}\right] + \frac{(\sigma_1 - \sigma_2)}{2}\left[1 + \frac{3a^4}{(a+x)^4}\right] + \frac{p_{\mathrm{I}}a^2}{(a+x)^2} \quad (2)$$

which, upon expansion to the first two terms in powers of x/a, becomes

$$p_{22}(x) = (\sigma_1 - 3\sigma_2 + p_{\mathrm{I}}) - (5\sigma_1 - 7\sigma_2 + 2p_{\mathrm{I}})\frac{x}{a} \quad (3)$$

$$= \sigma_{\mathrm{L}} - \sigma_{\mathrm{G}}\frac{x}{a} \quad (4)$$

with $\sigma_{\mathrm{L}} = (\sigma_1 - 3\sigma_2 + p_{\mathrm{I}})$ being the local stress at the cavity surface and $\sigma_{\mathrm{G}} = (5\sigma_1 - 7\sigma_2 + 2p_{\mathrm{I}})$ being associated with the stress gradient in the immediate vicinity of the cavity surface.

Intuitively, we expect that if crack nucleation occurs under conditions where the cohesive-zone size is small compared with the cavity radius, then this situation can be simulated in terms of a cohesive zone emanating from the planar surface of a semi-infinite solid, with the "applied stress" distribution being the linear-stress distribution immediately ahead of the actual cavity surface; the viability of this approach has been vindicated elsewhere [7]. Thus, consider the situation where a cohesive zone emanates from the planar surface of a semi-infinite solid (Fig. 2). It is assumed that the tensile stress along the plane $X_2 = 0$ in the absence of the cohesive zone is given by expression (4), with x being measured from $X_1 = 0$ along the X_1 axis; this stress simulates the tensile stress ahead of the cavity surface. If s is the cohesive zone size, the condition for finiteness of stress ($= p_{\mathrm{c}}$) at the leading edge of the cohesive zone within which the tensile stress has the uniform value p_{c}, is [8]

$$1.12(\sigma_{\mathrm{L}} - p_{\mathrm{c}}) - \frac{2.14\sigma_{\mathrm{G}}s}{\pi a} = 0 \quad (5)$$

while the relative displacement (crack nucleation) condition at $x = 0$, i.e. $v_{\mathrm{T}} = v_{\mathrm{c}}$, is given by the relation

$$v_{\mathrm{c}} = \frac{5.83s(\sigma_{\mathrm{L}} - p_{\mathrm{c}})}{E_{\mathrm{o}}} - \frac{1.77s^2\sigma_{\mathrm{G}}}{E_{\mathrm{o}}a} \quad (6)$$

where $E_{\mathrm{o}} = E/(1 - v^2)$, E being Young's modulus and v being Poisson's ratio. In relations (5) and (6), σ_{L} and σ_{G} are given by the expressions $\sigma_{\mathrm{L}} = (\sigma_1 - 3\sigma_2 + p_{\mathrm{I}})$ and $\sigma_{\mathrm{G}} = (5\sigma_1 - 7\sigma_2 + 2p_{\mathrm{I}})$ - see the comments after relation (4). Elimination of s between relations (5) and (6) gives

$$\frac{E_{\mathrm{o}}v_{\mathrm{c}}}{p_{\mathrm{c}}a} = \frac{4.79(\sigma_{\mathrm{L}} - p_{\mathrm{c}})^2}{\sigma_{\mathrm{G}}p_{\mathrm{c}}} \quad (7)$$

while the cohesive-zone size s at crack nucleation is given by relation (5) as

$$\frac{s}{a} = \frac{1.64(\sigma_{\mathrm{L}} - p_{\mathrm{c}})}{\sigma_{\mathrm{G}}} \quad (8)$$

Now, with the DBCS representation of a cohesive zone, the fracture toughness K_{IC} associated with the extension of a crack under LEFM conditions is given by the expression

$$K_{\mathrm{IC}} = [E_{\mathrm{o}}p_{\mathrm{c}}v_{\mathrm{c}}]^{1/2} \quad (9)$$

whereupon relation (7) can be written as

$$\frac{K_{\mathrm{IC}}^2}{p_{\mathrm{c}}^2a} = \frac{4.79(\sigma_{\mathrm{L}} - p_{\mathrm{c}})^2}{\sigma_{\mathrm{G}}p_{\mathrm{c}}} \quad (10)$$

and this is the crack-nucleation criterion. Because the derivation of the criterion has been based on the "applied" stress distribution in the immediate vicinity of the cavity surface, relation (10) is applicable only for the case where the cohesive zone size is small compared with the cavity radius, i.e. s/a is small or (see relations (8) and (10)) $(\sigma_L - p_c)$ is small compared with p_c, and so $K_{IC}^2/p_c^2 a$ is small.

Let us now consider special cases. First of all consider the case where the external stresses are $\sigma_1 = \sigma_2 = \sigma$. The cavity is internally pressurized, and there is a plane of weakness along the 1 axis so that fracture occurs preferentially along this plane. For this situation, noting that σ_L is then equal to $(p_I - 2\sigma)$, σ_G is equal to $2(p_I - \sigma)$ and that the nucleation theory is valid for small $(\sigma_L - p_c)$, the nucleation criterion (10) shows that crack nucleation occurs when the internal pressure in the cavity is raised to a value given by the expression

$$\frac{p_I}{p_c} = 1 + \frac{2\sigma}{p_c} + \frac{0.65 K_{IC}}{p_c a^{1/2}} \left[\frac{\sigma}{p_c} + 1 \right]^{1/2} \qquad (11)$$

This expression clearly highlights the limitations of a simple strength criterion for crack nucleation, which would be $\sigma_L = (p_I - 2\sigma) = p_c$ or

$$\frac{p_I}{p_c} = 1 + \frac{2\sigma}{p_c} \qquad (12)$$

because of the existence of the cavity size dependent term in relation (11). This term stems from the stress gradient ahead of the cavity surface and is responsible for a strengthening effect, which becomes more prominent as the cavity size decreases. Relation (11) also highlights the beneficial effect of the confining pressure σ with regard to crack nucleation; thus the relation shows that p_I increases as σ increases, a result which is not surprising and is consistent with both the numerical predictions and granite experimental results obtained by Hashida and co-workers [2].

Now consider the case where the external stresses are $\sigma_1 = \sigma$ and $\sigma_2 = 0$, and there is no internal pressure within the cavity. For this situation, σ_L is equal to σ and σ_G is equal to 5σ. Again noting that this paper's crack-nucleation theory applies to the situation where $(\sigma_L - p_c)$ is small, the nucleation criterion (10) shows that crack nucleation occurs when the applied stress σ attains a value which is given by the expression

$$\frac{\sigma}{p_c} = 1 + \frac{1.02 K_{IC}}{p_c a^{1/2}} \qquad (13)$$

As with the first case, this expression also highlights the limitations of the simple strength criterion for crack nucleation, which would be $\sigma_L = \sigma = p_c$; the difference between relation (13) and $\sigma_L = p_c$ is again due to the existence of the cavity-size dependent term in relation (13), which again is due to the stress-gradient effect. Relation (13) clearly shows that the applied stress σ required for crack nucleation increases as the cavity size decreases, a prediction that is in accord with experimental results on rock-like materials obtained by Lajtai and co-workers [9, 10], who refer to a fracture parallel to the loading axis as a primary fracture.

3. Discussion

On the basis of an idealised DBCS cohesive-zone description for a quasi-brittle material, the paper has formulated a very simple criterion (relation (10)) for the nucleation of a crack at the surface of a circular cylindrical cavity in an infinite solid. A planar surface simulation procedure has been used to facilitate the analysis, and consequently the theory is strictly applicable only to the situation where the cohesive-zone size at crack nucleation is small compared with the cavity size. The crack-nucleation criterion is expressed in terms of the internal pressure within the cavity, the stress system in the absence of the cavity, the cavity size and the cohesive-zone material characteristics.

Since the nucleation criterion has been expressed in a particularly simple form, it is easy to see how these various factors input into the nucleation criterion. For example it is easy to see how they are responsible for a flaw-strengthening effect, whereby the effective failure stress at the cavity surface is greater than the tensile fracture stress of the cohesive material, and how the degree of strengthening is affected by the stress gradient ahead of the cavity surface and hence its size, and also the applied-stress system. This importance of the stress gradient has been recognized by other workers, for example by Lajtai and co-workers [9].

Before closing this discussion, it is worth mentioning that although this paper has been concerned with the modelling of the behavior of a quasi-brittle material in a situation where there is a "tensile"-cohesive zone, similar considerations will also be relevant to situations where there is a "compressive" zone, i.e., failure due to a compressive circumferential stress at the cavity surface, referred to as a slabbing fracture.

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Received 17 June
and accepted 15 July 1998